Center for Algebraic Thinking

MODULE

Modeling: Translating Words to Equations-Distinguishing Relevant Variables and Maintaining their Meaning

BACKGROUND

This module is focused on the teaching of algebra translation as it appears between words and equations. Almost all word problems begin with a problem statement in written or spoken language that requires a semantic understanding of the problem situation and task. In algebra, there is usually a translation of that understanding into the "language" and syntax of algebraic expressions and equations. There is a related, although cognitively different, skill that is required to understand the semantics of a quantitative model that is derived from the interpretation of an algebraic expression or equation. These are the areas of interest for this module, and as they relate to model building in students learning algebra.

What's the important math?

The power of algebra is that it may be used to model quantitative relationships represented in equations that represent a very clever tool for answering quantitative questions that may shortcut far lengthier guess and check routines using arithmetic. The utility of algebra depends on the validity of the algebraic equation models and the reliability of their interpretations based upon a) an understanding of the meaning of the symbols used, and b) the correct application of the rules and procedures used in the "reduction and comparison" of algebraic expressions in equations. This module considers research on the learning and teaching of the former consideration, namely the meaning of the symbols used and the syntax of algebra as related but distinct from our natural language, particularly vernacular English.

The modules on ***Translating Words to Equations*** treat five fundamentally important ideas in the teaching and learning of algebra translation:

1) use of letters to signify *quantities* that may be:
    a) letter as unknown as in $5n = 2(n-3) + 8$
    b) letter as variable as in $5n = y/8$
2) numbers as *constants* as in 'a' above (the number 8)
3) numbers as *factors* that imply operations (as in the 5 in 5n)
4) use of the *equal sign* as a statement of precise numerical equivalence only.
5) variables of interest --- the need for determination of which variables matter and the avoidance of irrelevant and extraneous variables and maintaining their definitions after algebraic transformations of equations

Finally, we consider instructional suggestions intended to overcome extremely resilient misconceptions and offer problems that may promote class discussion for successful learning of algebra translation tasks.

1) **<u>SET</u>: Engage with a problem or problems that help teachers consider students' algebraic thinking (teachers' prior knowledge)**
Read the student description below. What, if anything, is wrong with the argument or with the conclusion?

Books and CDs Problem:

I went to the dollar store and bought the same number of books as CDs. Books cost two dollars each and CDs cost six dollars each. I spent $40 altogether. Assuming that the equation 2B + 6C is correct, what is wrong, if anything with the following reasoning? Be as detailed as possible.

$$2B + 6C = 40$$

Since B = C, I can write

$$2B + 6B = 40$$

$$8B = 40$$

This last equation says 8 books is equal to $40, which means that one book costs $5.

Read the student description below. What, if anything, is wrong with the argument or with the conclusion?

Given the following problem: "Using the letters 'S' for the number of students and 'P' for the number of professors, write an equation to represent the following relationship: 'At this university, there are six times as many students as there are professors'", a student wrote: 6S + P = T, and stated, "There are six students for each professor in the total."

2) **<u>STUDENTS</u>: Watch video clips of students describing their thinking as they engage with problems**
What do you learn from what you are hearing or seeing regarding students' thinking?

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Faulty algebraic thinking can occur in many phases of the problem solving process. What would a correct formulation look like?

3) **<u>RESEARCH</u>: Examine/discuss research (encyclopedia entries)**

In all mathematical problem-solving, understanding the problem and identifying relevant information are critical to success. Similarly, algebra translation tasks require that students identify the relevant variables so that an algebraic model can be formed into an equation that answers the question posed. It is also critical that the meanings of the relevant variables remain consistent during and after algebraic transformations of the original equations. These skills are especially important when students study systems of equations, but it first appears in the translation of words to equations in linear models involving two variables as in the examples above.

Identifying Relevant Variables:

The 'Students and Professors' problem is almost as simply stated as it could be with only two variable quantities in the statement. Nevertheless, many students will identify a third term that could be a number as we saw earlier (S + 6 = P), or a letter that could be used as another variable of interest to them, although not relevant to the solution of the problem.
Many students will write the following solution to the problem: 6S + P = T, and state, "For every six students there is one professor and they make up the total people on campus." The introduction of the total people on campus is not requested or required to represent the relationship between the number of students and professors. But for the students who invoke the total number on campus, there is no apparent relationship of the parts without consideration of the whole. Furthermore, for the equation to be correct, these students feel that the whole must appear in it.

Another extraneous variable of interest in this problem is introduced when students consider the proportional relationship as a number of groups. For example, when asked to use the equation 6S = P to predict one value from another or to establish a table of data, many students invent another variable, 'x', which they define as "the number of groups of six students and one professor. They adjust their equation accordingly, and write, x(6S = P), where the 'x' multiplies the entire equation.

Maintaining the Definition of Variables:

The confusion that students demonstrate about the meaning of the letters in their equations is further exacerbated when algebraic transformations are applied to their equations. One study used the following problem to document the too fluid designation of meaning to symbols in algebra:

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2B + 6C = 40

Since B = C, I can write

2B + 6B = 40

8B = 40

This last equation says 8 books is equal to $40, which means that one book costs $5.

This problem generates some excellent discussion from groups of students, since it clearly poses an interpretation that contradicts an earlier statement, namely that a book started out costing $2 each, but somehow increased to $5 each. The challenge is for students to see the requirement that B and C represent the number of books and CDs throughout the problem, and even after the substitution of B for C since their numbers are equal. The meaning of the variables is often over-generalized to become something to do with books and CDs, and seems to shift as algebraic transformations alter the equation's appearance.

4) **<u>ASSESSMENT</u>: Consider assessments (Formative Assessment Database)**

Here is a recreation of a discussion that the author had with a student who had written the reversed equation for the following problem:

Feet & Yards Problem:

> Write an equation that describes the relationship between the number of feet and the number of yards in any measure of length. Use "F" for the number of feet and "Y" for the number of yards. [Imagine you go out to a football field and measure some length. That length could be stated as a number of feet or a number of yards. What is the equation that describes the relationship between those two quantities.]

The student wrote, 3f = y, and said, "There are three feet in every yard." When asked if the equation could be used to find the number of feet in 30 yards, the student replied:
"Sure, this equation says there are three feet in one yard. In algebra, you don't have to put the 1 but you could. [She wrote, 3f = 1y] So this says there are three feet in one yard, but you don't have one yard, you want ten yards, so you would have to multiply the 1y by ten [student writes 3f = (10)1y], but in algebra what you do to one side of the equation, you have to do to the other side too, so I am going to multiply the 3f by ten too [student writes (10)3f = (10)1y]. When I multiply everything, you get 30f = 10y, or there are thirty feet in ten yards."

Describe how you would create an intervention that will help this student to understand that although her equation is a successful model for solving the problem, it is not quite the right equation for algebraic use in this problem.

Airlift Problem:

> You are organizing a Red Cross airlift center to aid earthquake victims. Planes carrying crates of food can make several round trips per day to an earthquake area. Write an algebraic equation in the form of: P = ________., that will predict the number of planes (P) needed to deliver C crates of food to an earthquake area in D days. Be sure to specify your variables clearly. Use the following variables:
>
> P = the number of planes needed
> N = the number of food crates each plane can carry in one trip
> C = the Total number of food crates needed in the area each day
> T = the number of round trips a plane can make in one day
> D = the number of days you need to bring food to the area

Rent Problem:

> Michelle and Lisa share an apartment, but since Lisa earns a higher salary than Michelle, they do not split the rent evenly. Instead, each woman pays the same percentage of her income toward rent. Write an equation that can be used to calculate the percentage of their income that will be paid toward rent.

5) <u>SUGGESTIONS FOR TEACHING</u>: **Consider strategies based on research (including apps)**

1) Encourage students to write the definition of variables for the problem by beginning with the phrase, "let n = the number of …"
2) To avoid confusing the variable with a label, use letters that have no connection to the words in the problem.
3) Make a table of data.
4) Draw a picture or diagram of the problem situation.
5) Make a graph of the functional relationship and relate the graph to the equation.
6) Write a thought-process as the equation is created
7) Check the final equation with a pair of numbers that are known to satisfy the given problem statement.
8) Write a computer program that inputs one value and outputs the other.
9) Engage in a discussion where mental arithmetic is used to calculate the output of an input value

What follows below are some problems that may be posed to foster teaching and learning of algebra translation tasks. The solution of these problems, and the extended discussions and debate around them will help students and teachers to discover the ideas discussed here.

Algebra Translation Tasks for Instruction:

Translate to algebraic statements:

"**Jim's marbles plus five is equal to the same number of marbles as Lucy has."**

[That is an odd statement: How about:]

"If Jim had five more marbles than he has, then he would have as many marbles as Lucy has."

"Leslie weighs 10 pounds more than Janet."

"Leslie weighs 10 pounds more than Janet, and Leslie weighs 137 pounds."

"I am just as rich as my neighbor."

"I am just as eloquent as my neighbor."

"I am much more handsome than my neighbor is."

"I am three times more handsome than my neighbor is."

"I am 24 years younger than my father."

"My father is twice as old as I am."

"At this school, there are 20 times more students than there are teachers."

"In the school cafeteria, for every four people who drink chocolate milk five people drink white milk."

"In 13 years Ellen will be two-thirds of her father's age then."

"Write an equation that relates the number of feet to the number of yards in any measure of length."

"Write a mathematical equation that relates the number of weeks in "x" years to the number of months in "x" years."

"In a biathlon an athlete swam S miles in Y hours and ran R miles in 5 times the amount of time it took him to swim. Write an expression for his average speed."

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6) **<u>Did the preservice teachers understand? How do you know? Evidence</u>**

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See Algebraic Thinking Encyclopedia at http://algebraicthinking.org/algebra-thinking-references#Modeling for additional references.