Center for Algebraic Thinking

MODULE

Modeling: Translating Words to Equations-The Role of Numbers in Equations

BACKGROUND

This module is focused on the teaching of algebra translation as it appears between words and equations. Almost all word problems begin with a problem statement in written or spoken language that requires a semantic understanding of the problem situation and task. In algebra, there is usually a translation of that understanding into the “language” and syntax of algebraic expressions and equations. There is a related, although cognitively different, skill that is required to understand the semantics of a quantitative model that is derived from the interpretation of an algebraic expression or equation. These are the areas of interest for this module, and as they relate to model building in students learning algebra.

What’s the important math?

The power of algebra is that it may be used to model quantitative relationships represented in equations that represent a very clever tool for answering quantitative questions that may shortcut far lengthier guess and check routines using arithmetic. The utility of algebra depends on the validity of the algebraic equation models and the reliability of their interpretations based upon a) an understanding of the meaning of the symbols used, and b) the correct application of the rules and procedures used in the “reduction and comparison” of algebraic expressions in equations. This module considers research on the learning and teaching of the former consideration, namely the meaning of the symbols used and the syntax of algebra as related but distinct from our natural language, particularly vernacular English.

The modules on ***Translating Words to Equations*** treat five fundamentally important ideas in the teaching and learning of algebra translation:

1) use of letters to signify *quantities* that may be:
    a) letter as unknown as in $5n = 2(n-3) + 8$
    b) letter as variable as in $5n = y/8$
2) numbers as *constants* as in ‘a’ above (the number 8)
3) numbers as *factors* that imply operations (as in the 5 in 5n)
4) use of the *equal sign* as a statement of precise numerical equivalence only.
5) variables of interest --- the need for determination of which variables matter and the avoidance of irrelevant and extraneous variables and maintaining their definitions after algebraic transformations of equations

Finally, we consider instructional suggestions intended to overcome extremely resilient misconceptions and offer problems that may promote class discussion for successful learning of algebra translation tasks.

1) **<u>SET</u>: Engage with a problem or problems that help teachers consider students' algebraic thinking (teachers’ prior knowledge)**

Write equations to symbolize the relationships in the following statements:

> "If Sally had 5 more dollars than what she has, then she would have the same amount of money that Wendy has."
>
> "If Sally's bank account was multiplied by 5, it would contain the same amount of money as Wendy's bank account."
>
> "Using the letters 'S' for the number of students and 'P' for the number of professors, write an equation to represent the following relationship: 'At this university, there are six times as many students as there are professors'"

2) **<u>STUDENTS</u>: Watch video clips of students describing their thinking as they engage with problems**
What do you learn from what you are hearing or seeing regarding students' thinking?

See Module 1.

The same problem sequence can be employed as was used in the previous module. This time follow up questions would elicit reasons for choice of numbers included in the algebraic sentences.

3) **<u>RESEARCH</u>: Examine/discuss research (encyclopedia entries)**
While there are equations that are written with no numerals, and may even include letters that stand for 'constants', most students see and expect equations that contain numerals but are confused as to their meanings and how to interpret their syntax when writing equations. This is sometimes observed in student interpretations of "y = mx + b" where the y-intercept 'b' and the slope 'm' are sometimes referred to as constants, but almost always represented with numbers. Here are two important confusions that students experience with numbers in algebra translation tasks.

Numbers as Constants

Most students learn about the use of numbers as constants in a context where the numbers and the letters make obvious connections to algebra equations. For example when students are asked to symbolize the relationship in the statement:

> "If Sally had 5 more dollars than what she has, then she would have the same amount of money that Wendy has."

The equation S + 5 = W is a simple direct translation from words (however awkward from natural language) to equations where the use of the number 5 works. However, in the 'Students and Professors' problem above, some students write, S + 6 = P, where they state, "There are six more students than professors". Here again, their semantic understanding may be confused but not always. Students have used the letters here to mark two columns on a table where the ratio image of students and professors lining up 6 to 1 is an accurate image of the meaning in the problem, although the syntax of their equation would infer otherwise; the table for S + 6 = P would look very different for a mathematician.

Numbers as Factors that Imply Operations

In an algebra equation, when a number is placed alongside a letter (either before or after, but customarily before), mathematicians interpret that the quantity signified by the letter is to be *multiplied* by the number; the number is a *factor* whose operation is implied. Analogous to the awkwardly phrased translation task with Sally and Wendy above, a words-to-equations task could be written explicitly as:

> Write an equation to symbolize the relationship in the following statement:
>
> "If Sally's bank account was multiplied by 5, it would contain the same amount of money as Wendy's bank account."

This problem is so obvious as to result in a nearly universally correct response from any students who understand the implication of the operation for multiplication in the equation: 6S = W.

In the 'Students and Professors' problem, the equation, 6S = P, would suggest to a mathematician that the number of students would need to be multiplied by 6 to equal the number of professors. This is the reverse of the situation presented in the problem, and the correct version would be S = 6P where the number of professors would be multiplied by 6 to yield the correct number of students. The same issue was observed in the 'centimeters and meters' problem above.

The use of '6S' is often interpreted as "six students" rather than "six times the number of students". As such, the 6 in '6S' is used as a modifying adjective rather than as a call for action to multiply (verb). Some students have defended their use of 6S = P by drawing a large circle for the students with the label '6S' inside and alongside a smaller circle for the professors with the label 'P'. They state, "The S gets the 6 because it is six times bigger than Professors." Here the number is described as the defense for the action of drawing the circle of students larger than the professors which seems to describe the circle rather than the number of students. Again, the number is used as a descriptive modifier for a set of students. Alternatively, the number may be viewed as an operator as it "blows up" the size of the illustration of the number of students.

Finally, there is an obvious link between the language of the problem and the connection of the factor 6 to the letter S, and that is the phrase, "six times as many students as professors". Students who indeed understand the operational implication of the number 6 as a factor implying multiplication often read the phrase as "six times students". In this case, they understand both the quantitative relationship of the given problem situation and the symbolic significance of numbers next to letters in equations. However, they have not connected the operation in a way that would reflect the desired quantities given the relationship between the variables.

The Meaning of the Equal Sign

There are many uses and misuses of the equal sign in mathematics that will not all be described here. The most widely accepted use in algebraic equations is an equivalence of numerical quantity. Even with this definition, there are other meanings and interpretations that may be mathematically and numerically equivalent and yet cognitively quite different. For example: the equations, 6P = S, S = 6P, S/6 = P, P/S = 1/6, etc. may all be viewed as transformations of the same equation that all maintain numerical equivalence. Yet, for students who describe these equations, they each represent cognitive models that are distinct from one another.

As we will see, the use of the equal sign presents further problems for students both for how the term "equal" is used in natural language, and for the confabulation of multiple use of the term in mathematics and in the sciences. For example, text books and other references often state units-conversions with equal signs and with letters that signify units, as we saw in the 'centimeters and meters' problem. Some of these confusions will be described below.

Confusion of Equal sign for "As"

In the statement: "6 times as many students *as* professors", the second "as" is referred to as being the same as "equals". Hence, students will write 6S = P. In fact, students have experience in mathematics education with just such a use of the equal sign when we teach them the algebraic representation of proportions: "a/b = c/d is often recited as "a is to b *as* c is to d". There are many examples where students, when asked for an equation in the students and professors problem, will respond with 6S:P and use the colon mark as their proxy for an equal sign. To them, this is an equation of sorts, and the colon is a satisfactory if not synonymous representation for equality. The same occurs when students use the vinculum as in 6S/P as their presentation of a final equation.

Mistaken use of Equal sign as an Assignment ---"for every"

In the 'students and professors' problem, students will often interpret the equal sign in the equation 6S = P as "for every" six students there is one professor. In this case the equal sign is used as a type of assignment; six students are assigned to one professor. Another term to describe this type of equality could be "correspondence" in as much as the number of students seems to correspond to the number of professors. While these distinctions may appear trivial, they may often manifest as powerful cognitive structures for the students who think in a particular way.

4) **<u>ASSESSMENT</u>: Consider assessments (Formative Assessment Database)**

Write equations to symbolize the relationships in the following statements:

"In 13 years Ellen will be two-thirds of her father's age then."

"Write a mathematical equation that relates the number of weeks in "x" years to the number of months in "x" years."

"In a biathlon an athlete swam S miles in Y hours and ran R miles in 5 times the amount of time it took him to swim. Write an expression for his average speed."

5) **<u>SUGGESTIONS FOR TEACHING</u>: Consider strategies based on research (including apps)**

1) Encourage students to write the definition of variables for the problem by beginning with the phrase, "let n = the number of …"
2) To avoid confusing the variable with a label, use letters that have no connection to the words in the problem.
3) Make a table of data.
4) Draw a picture or diagram of the problem situation.
5) Make a graph of the functional relationship and relate the graph to the equation.
6) Write a thought-process as the equation is created
7) Check the final equation with a pair of numbers that are known to satisfy the given problem statement.
8) Write a computer program that inputs one value and outputs the other.
9) Engage in a discussion where mental arithmetic is used to calculate the output of an input value.

6) **<u>Did the preservice teachers understand? How do you know? Evidence</u>**

<u>REFERENCES</u>

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See Algebraic Thinking Encyclopedia at http://algebraicthinking.org/algebra-thinking-references#Modeling for additional references.