Center for Algebraic Thinking

<u>MODULE</u>

Analysis of Change: Connecting graphs to algebraic relationships

BACKGROUND
It is important for students to be able to construct a graph that represents an algebraic equation. Equally important is the ability to form an algebraic equation that represents a simple graph (such as a linear graph). This module is designed to help candidates understand some of the important skills and common misconceptions associated with connecting algebraic expressions to graphical representations.

1) **<u>SET</u>: Engage with a problem or problems that help teachers consider students' algebraic thinking (teachers' prior knowledge)**

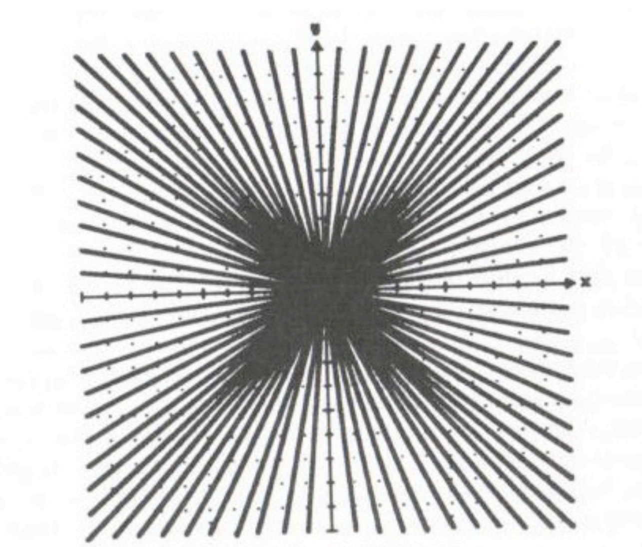


FIG. 4.4. A Starburst.

Task: Consider the starburst pattern above. Using a graphing utility (such as grapher or a graphing calculator), recreate the starburst pattern.*

Mathematical Objective: The expectation is that preservice teachers should have the ability to recognize that the lines in the starburst pattern are of the form y=mx. However, students must think deeply about slope in order to have all of the lines evenly spaced. (For example, integer values for “m” do not result in the above figure.)

2) <u>**STUDENTS**</u>**: Watch video clips of students describing their thinking as they engage with problems (these video clips must still be developed/inserted)**

1. Graph y = 2x+1. Tell me your thinking as you create the graph.

    Follow-up questions:

    - Why did you decide to have the graph go through this point? (point out the y-intercept)
    - How did you know the slope of this line?

    Possible Guiding Questions (if needed):

    - Suppose x = 0. Show me where x = 0 on the graph.
    - In the equation, if x=0, what does y equal?
    - What do you remember about y=mx+b form?
    - Would it help to make a table?

2. What would happen if you change the 1 to a 7? What would happen to the graph? Why do you think that?

3. Graph y = 2x+7.

    (Refer to previous guiding questions if needed.)

4. Write the equation of a new line halfway between the lines you drew.

    Follow-up questions:

    - Explain what it means for the new line to be half-way between the lines?
        - What should the slope of the new line be to lie between the two lines?
        - What should the y-intercept of the new line be to lie between the two lines?
        - What should the x-intercept of the new line be to lie between the two lines?
    - Graph the new line that is between the other two lines.

- These questions are aimed at uncovering how students think about how the value of “b” will affect the graph of a line. It is also aimed to uncover the relationship between the graphical representation of parallel lines and the algebraic representation of the slope of those lines.

3) <u>**RESEARCH**</u>**: Examine/discuss research (encyclopedia entries)**

Research indicates that functions can be thought of as both a process and an object, and both types of conceptions are important for understanding graphing and equations of linear functions. The process conception of function regards functions as an input-output machine: if I plug in one value of x, I get out a different value of y. This conception is important because it allows students to use equations to plot points. In addition, this conception helps students to recognize that the graph of a line is actually a set of individual points, each of which are individual solutions which satisfy the algebraic equation. The object conception linear functions allows students to think of the entire function as a single object. As a single object, students can think about translating the line with vertical or horizontal translations and can identify holistic properties of the line such as “slope.” For example, a student with an object conception functions would be able to graph the equation y=2x+3 by taking the graph y=2x and shifting it vertically three units. See entries on **Analysis of Change: Action/Process/Object conception of function.**

Research indicates that students tend to think of changes in the “b” value of “y=mx+b” as a horizontal shift rather than a vertical shift. This intuition is not incorrect: lines of the same slope can be thought of as both horizontal and vertical shifts of one another. However, it is important for students to understand that a one unit increase in the value of “b” results in a vertical shift of one unit, whereas the same one unit shift in “b” results in a horizontal shift of -b/m units. (See entry on **Analysis of Change: Shifting Linear Graphs).**

In addition, research indicates that students have trouble with the fact that the standard linear equation “y=mx+b” does not explicitly list the value of the x-intercept. When writing equations for linear functions, students improperly place the value of the x-intercept into the standard equation of the line. (See entry on **Analysis of Change: The Missing x-intercept Value in “y=mx+b”)** Although teachers frequently disregard this as a trivial error, the relationship between the x-intercept and the values of “m” and “b” is important.

4) Formative **ASSESSMENT: Consider assessments (formative assessment database)**
What are some of the possible values for the slope of the line that lies in the shaded region? How do you know?*

a. Graph the equation y=x+4.
Jimmi said that this line would go through the axis at (4,0) because in the equation you add 4 to x. Do you think that Jimmi was right?**

b. Graph the equation y = x and then change it to the equation y = x+3. Predict how the graph would change.**

5) **SUGGESTONS FOR TEACHING: Consider strategies based on research (including apps)**

Graphing calculators can be an effective tool for allowing students to quickly graph multiple functions on the same set of axis. By varying the values of "m" and "b" students can explore how the coefficients in the algebraic equation affect the graphical representation of the function. Graphing calculators provide linked approaches (numeric, graphical, and algebraic) to the same problem and allow complex math to become more accessible to a greater number of students.

Apple's Grapher program (Included in the Utilities folder of the Apple OS X 10.4 and later) can also be an effective tool for this along with Web-based applications like Graph Sketcher by Shodor Interactivate (Use Apple Safari or Windows Internet Explorer for best results with this Java-based site).

To understand the relationship between algebraic expressions and their graphs, it is best to start with a simple case. Students should explore the case where m=1, and see the connection between how the changing value of "b" affects the x-intercept. Next, students should explore cases where the slope is not 1. As the value of the slope changes, students can see how the shift in "m" relates to the steepness of the slope of the graph, and a shift of the x-intercept toward the origin. This guides students away from seeing the x-intercept as a reflection of a change in "b" and seeing it as also depending on "m."

REFERENCES

Hennessy, S., Fung, P., & Scanlon, E. (2001). The role of the graphic calculator in mediating graphing activity. International Journal of Mathematical Education in Science and Technology, 32(2), 267-290.

*Moschkovich, J., Schoenfeld, A., & Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T. A. Romberg, E. Fennema & T. P. Carpenter (Eds.), Integrating research on the graphical representation of functions (pp. 69-100). Hillsdale, NJ: Lawrence Erlbaum Associates Publishers.

**Moschkovich, J. (n.d.). "Students" use of the x-intercept as an instance of a transitional conception. *Educational Studies in Mathematics*, *37*(2), 169.

Noble, T., Nemirovsky, R., DiMattia, C., & Wright , T. (2002). *On Learning to See: How Do Middle School Students Learn to Make Sense of Visual Representations in Mathematics?*, TERC; Annual Meeting of the American Educational Research Association (AERA), 2002, New Orleans, LA.

Penglase, Marina and Arnold, Stephen. (1996). The Graphics Calculator in Mathematics Education: A Critical Review of Recent Research. *Mathematics Education Research Journal, 1996, Vol8, No.1, 58-90*

Representation, Vision, and Visualization. Duval, Raymond. 1999. Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.

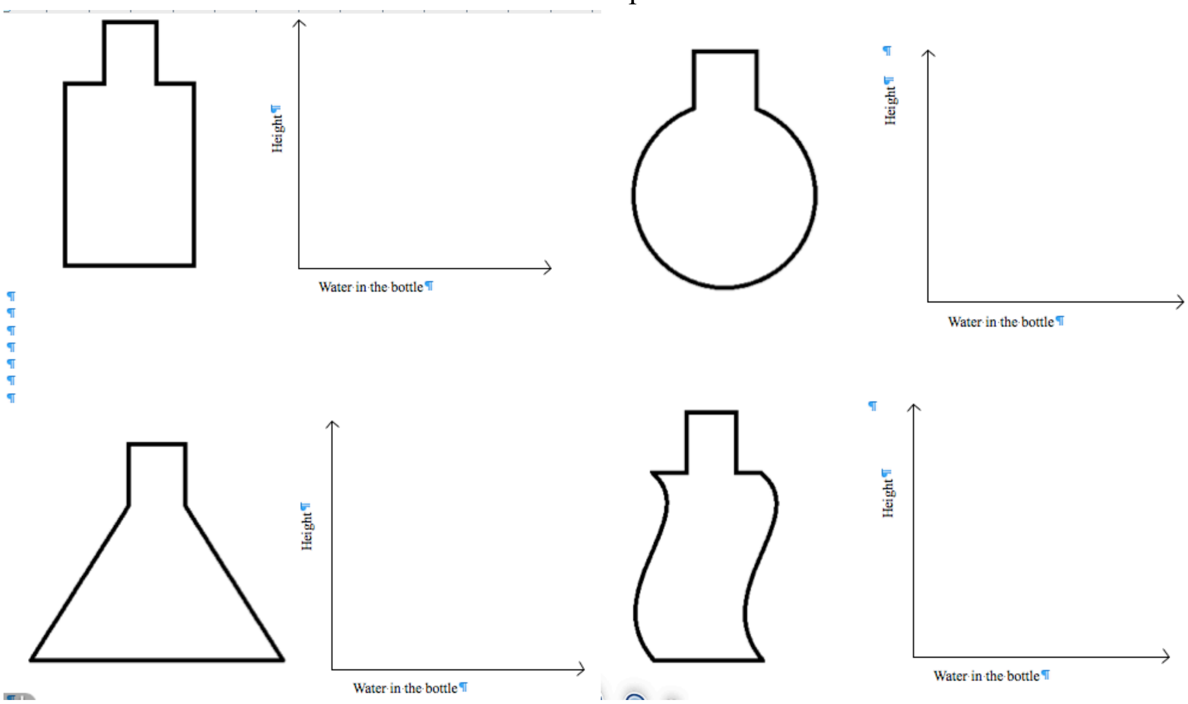
Graphical relationship of 2 changing variables--slope is not a number it is a relationship...(Understanding Slope document)

0) **Description:** Slope is a measurement of the relationship between two changing quantities. This is more than just "rise over run" or the value "m" in the equation y=mx+b. In this module candidates explore the changing ratio of two variables by examining a scenario and creating a graph.

1) **<u>SET</u>: Engage with a problem or problems that help teachers consider students' algebraic thinking (teachers' prior knowledge)**

Bottle problem (from Carolyn and Shona's class)
How is the rate of change related to the slope of line

Directions: Imagine these bottles filling with water. For each bottle, sketch a graph of the height of the water as a function of how much water has been poured into the bottle.



Compare your graph to a neighbor. Explain to each other why you drew each graph and discuss similarities and differences in the graphs that you drew. Compare each graph from when the bottle is empty to when it is half full. Compare each graph from when the bottle is half full to when the water reaches the neck of the bottle.

Notes for the instructor: Do candidates note that the graphs for the first and last bottles are the same (i.e. the "waviness" of the lines makes no difference to the overall height of the water in the bottle)? Focus candidates' attention on the variation in the shape of the bottle and how that variation affects the height of the water in the bottle. If candidates struggle to visualize the graphs, have them actually do the problem and measure the height to plot their graph (to do that you'll need different shaped bottles/glasses, cm ruler and measuring cups). Fixed amounts of water might be helpful. Possible bottles that are similar to the sketch: ketchup bottles, fancy water glasses, jars, olive oil bottles, salad dressing bottles--check your refrigerator/cupboard!

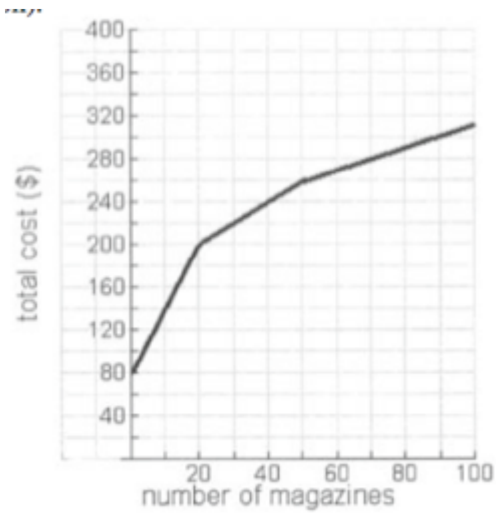
Answer key: Graphs for the bottle problem (*construct using an online tool or sketch and scan)*

2) **STUDENTS: Watch video clips of students describing their thinking as they engage with problems**
Film them solving the bottle problem, with the question being "which bottle goes with each graph?" but consider having them match a pre-drawn graph with the bottle instead of having them draw the graph. Key questions:
Why did you match the bottle you chose with the graph? If a student says, "I don't know" to the matching activity, ask them to measure the first couple of points. Did you notice that the graph is comparing the height of the water to the amount of water poured?

3) **RESEARCH: Examine/discuss research (encyclopedia entries)**
Research suggests that students have difficulty connecting slope with the idea of "rate of change." (consider matching cards that show bottles and graphs as an intermediate). Many students learn the mnemonic "rise over run," and can only think of slope in terms of geometric ratios (as I go up this much, I go over this much.) Students might also have an algebraic understanding of slope and be able to identify the slope of a line given a linear equation. However, having only one of these conceptions of slope is not sufficient. Students should be able to flexibly reason between both algebraic and geometric representations of slope, and they should also have a contextual understanding of slope: slope is a measurement of the relationship between two changing quantities.

Students will better understand the idea of rate of change if they explore it in more than one context. Rate of change and accumulation can be studied in motion and money contexts. In one study Algebra I students used motion detector technology and interactive banking software in a unit focused on slope, ratio, and rate of change. Interviews conducted with students at the end of the study indicate a surprising finding. Even if a person only partially understands rate of change in one context (i.e. motion), that does not mean that they cannot see the connection between the first context and a second (i.e. banking). Making the connection between contexts does *not* require full understanding of one of the contexts.

4) **ASSESSMENT: Consider assessments (formative assessment database)**
How would this learning be assessed? Maybe the magazine problem.
A school produces a school magazine once each year. When the magazines are printed there is an initial cost of preparing the magazine for printing and then a cost per magazine for the actual printing. Suppose Best Printers cost per magazine varies depending on the number of magazines printed as shown on the graph below:



How does the rate change from section A to Section B? From section to B to Section C. How do you know?

If you saw this graph in the water bottle problem, what shape would the glass be? How does your shape connect to the graph?

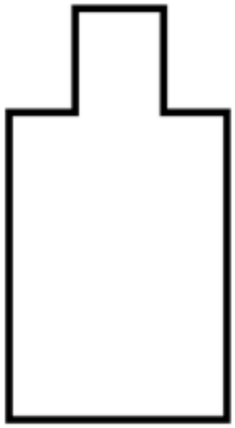
Researcher: If you saw that graph [Figure 7] in MathWorlds what would that tell us?
Can you describe how the elevator is moving?
Joe: It would go fast then slower and then go really slow.

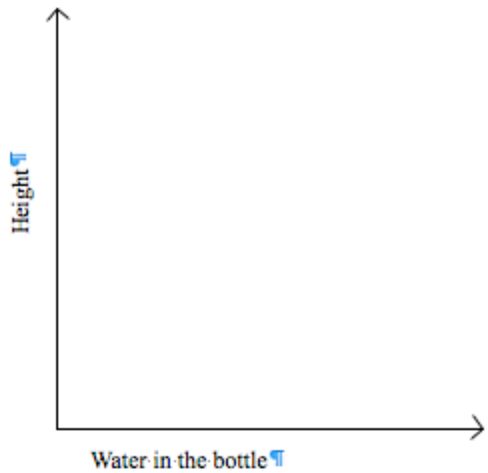
5) **INTERVENTION: Consider strategies based on research (including apps)**
Encourage students not just to solve the problems and plot graphs but to discuss their solutions with their classmates and with the teacher.

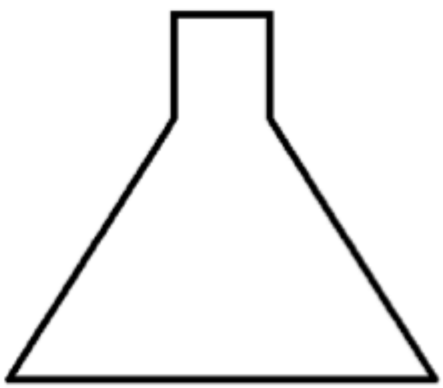
Back it up and and measure the height of the water and the corresponding quantity.

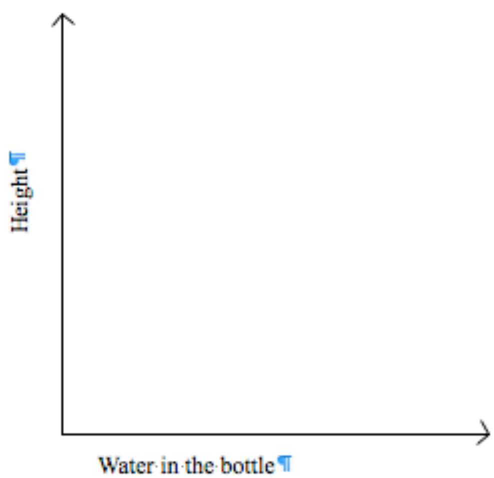
One of the unique things about these particular problems is that the two changing variables are the height of the water and the volume of water poured. Note that time is not one of the variables.

Existing online applets that are similar to this problem: **Technology**Create an applet that models the "bottle problem" (Carlson):
Imagine these bottles filling with water. Each time you pour in a fixed amount of water. Sketch a graph of the height of the water as a function of how much water has been poured into the

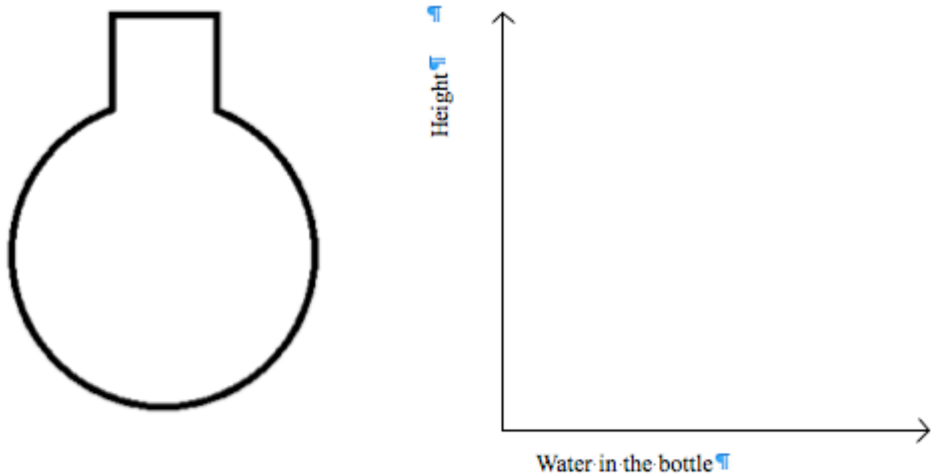


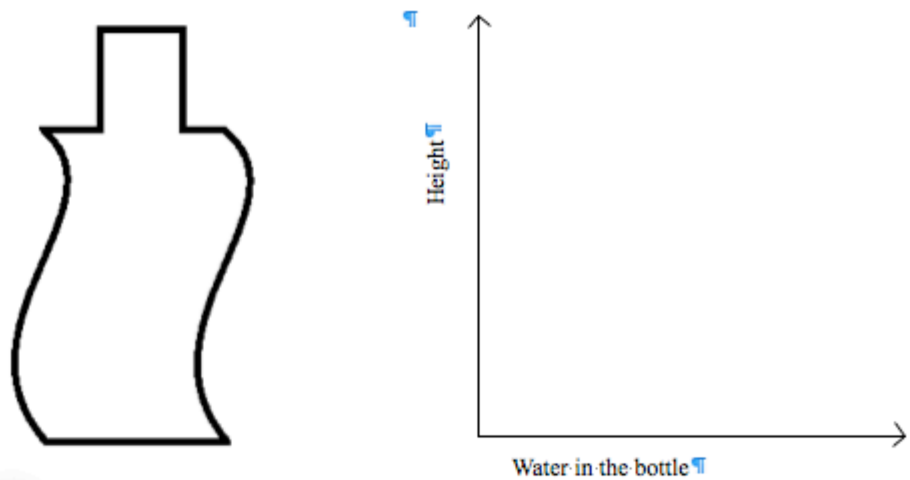






bottle.





This applet would be related to (but different from):
1) "Fill and Pour" in the NLVM.
http://nlvm.usu.edu/en/nav/frames_asid_273_g_4_t_4.html?from=category_g_4_t_4.html
2) "How high" in the NLVM http://nlvm.usu.edu/en/nav/topic_t_4.htm
3) "Flowing through mathematics" http://illuminations.nctm.org/ActivityDetail.aspx?ID=16

6) References:

Bottle problem: Carlson references

Magazine problem: Herbert, S., & Pierce, R. (2008). An 'Emergent Model' for Rate of Change. International Journal of Computers for Mathematical Learning, 13(3), 231-249. P. 140

Sample graphing module from Steve

0) **Description**

1) **SET: Engage with a problem or problems that help teachers consider students' algebraic thinking (teachers' prior knowledge)**
Three dimensional graph. (Apple Grapher) How do we know which is which axis? What one equation will help me see what each axis represents?

2) **STUDENTS: Watch video clips of students describing their thinking as they engage with problems**
Video of formative assessments below.

3) **RESEARCH: Examine/discuss research (encyclopedia entries)**
Students' understanding of the interaction between axes.

4) **ASSESSMENT: Consider assessments (formative assessment database)**

5) **INTERVENTION: Consider strategies based on research (including apps)**

Module template

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